Resit test 3 Numerical Mathematrics 2
April 13, 2023

Duration: 1 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. Suppose that $f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x)$, where ϕ_i, $i = 0, \cdots$, are orthogonal polynomials in some innerproduct.
   (a) [1] Show that $a_i = (f, \phi_i)/(\phi_i, \phi_i)$.
   (b) [1.5] Show that $\sum_{i=0}^{\infty} a_i^2 (\phi_i, \phi_i) = (f, f)$. What is the name of this expression and what does this expression mean?
   (c) [0.5] Assume $f(x) = p_n(x)$ where p_n is a polynomial of degree n. Show that $a_i = 0$ for $i > n$.
   (d) [0.5] What will be the error if we approximate $f(x) = p_n(x)$ by the first $n - 1$ terms of the expansion in orthogonal polynomials?
   (e) [0.5] Suppose $f(x)$ is defined on a finite interval. For which orthogonal polynomials ϕ_i will the approximation in the previous part give us the minimax approximation, i.e., the best polynomial approximation. Explain also why.
   (f) [0.5] Suppose that the coefficients a_i decrease rapidly with i. Based on the previous parts, show that the same orthogonal polynomials as in part e are the best choice to find least squares approximations to the minimax approximation?

2. Consider for arbitrary $f(x)$ the integral $\int_0^{\infty} \exp(-x) f(x) dx$.
   (a) [3.5] Using orthogonal polynomials, derive that for this integral the Gauss rule using one interpolation point is simply $f(1)$.
   (b) [1] Determine the degree of exactness of the rule in the previous part by applying it to $1, x, x^2, ...$, respectively. Does this comply with the theory?